

Average position in quantum walks with a U(2) coin

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We investigated discrete-time quantum walks with an arbitrary unitary coin. Here we discover that the average position $\langle x \rangle = \max(\langle x \rangle) \sin(\alpha + \gamma)$, while the initial state is $1/\sqrt{2}(|0L\rangle + i|0R\rangle)$. We prove the result and get some symmetry properties of quantum walks with a U(2) coin with $|0L\rangle$ and $|0R\rangle$ as the initial state.

I. INTRODUCTION

Quantum walks (QWs) were first introduced in 1993 [1] as generalization of classical random walks. According to the time evolution, QWs can be divided into discrete-time and continuous-time [2] QWs. Recently, both continuous-time [3] and discrete-time [4] QWs are found to be universal for quantum computation. A number of quantum algorithms based on QWs have already been proposed in [5–10]. In addition, QWs in graph [11], on a line with a moving boundary [12], with multiple coins [13] or decoherent coins [14] have been discussed also.

QWs using a SU(2) coin was introduced by Chandrashekar, *et al.* [15], where the standard deviation and measurement entropy properties were discussed. Here we discuss the symmetry and average position properties for the QWs with a U(2) coin.

II. HADAMARD QUANTUM WALKS

In this paper, we always discuss within the discrete-time QWs. The total Hilbert space for QWs is given by $\mathcal{H} \equiv \mathcal{H}_P \otimes \mathcal{H}_C$, where \mathcal{H}_P is spanned by the orthonormal states $\{|x\rangle\}$ and \mathcal{H}_C is the two-dimensional coin space spanned by two orthonormal states $|L\rangle$ and $|R\rangle$.

Each step of the QWs can be splitted into two operations: the evolution of coin state and the particle movement according to the coin state.

Here the Hadamard walk, the coin is evolved by applying the Hadamard operation:

$$H = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix},$$

the particle movement operator is given by

$$S = e^{ip\sigma_z} = \sum_x S_x, \quad (1)$$

where p is the momentum operator, σ_z is the Pauli- z operator,

$$\hat{S}_x = |x+1\rangle\langle x| \otimes |R\rangle\langle R| + |x-1\rangle\langle x| \otimes |L\rangle\langle L|. \quad (2)$$

Therefore, after t steps QWs

$$\begin{aligned} |\Psi_t\rangle &= [S(I_P \otimes H)]^t |\Psi_0\rangle \\ &= \left[\sum_x S_x(I_P \otimes H_C) \right]^t |\Psi_0\rangle, \end{aligned} \quad (3)$$

where $|\Psi_{in}\rangle$ is the initial state of the system.

III. GENERALIZED DISCRETE TIME QUANTUM WALKS

An arbitrary one-qubit unitary operation can be written as a U(2) matrix:

$$U_{\alpha,\beta,\gamma,\theta} = e^{i\theta} \begin{pmatrix} e^{i\alpha} \cos \beta & -e^{-i\gamma} \sin \beta \\ e^{i\gamma} \sin \beta & e^{-i\alpha} \cos \beta \end{pmatrix} \quad (4)$$

For example, the Hadamard operator can be described in the form $H = U_{\frac{\pi}{2},\frac{\pi}{4},\frac{\pi}{2},-\frac{\pi}{2}}$. By replacing the Hadamard coin with an operator $U_{\alpha,\beta,\gamma,\theta}$, we can obtain the generalized QWs [15], which can be written as

$$|\Psi_t\rangle = [S(I_P \otimes U_{\alpha,\beta,\gamma,\theta})]^t |\Psi_0\rangle \quad (5)$$

Lemma 1. *The quantum walks have the same probability distribution with a U(2) coin or a SU(2) coin which has the same parameters α , β and γ in the U(2) matrix.*

Proof. The SU(2) coin operator can be written as

$$U_{\alpha,\beta,\gamma}^S = \begin{pmatrix} e^{i\alpha} \cos \beta & -e^{-i\gamma} \sin \beta \\ e^{i\gamma} \sin \beta & e^{-i\alpha} \cos \beta \end{pmatrix}, \quad (6)$$

Then the U(2) coin $U_{\alpha,\beta,\gamma,\theta} = e^{i\theta} U_{\alpha,\beta,\gamma}^S$. The probability distribution for QWs with a U(2) coin after t steps:

$$\begin{aligned} P(x,t) &= |\langle x|\Psi_t\rangle|^2 = |e^{it\theta}\langle x|\Psi_t^S\rangle|^2 \\ &\equiv P^S(x,t), \end{aligned} \quad (7)$$

where $|\Psi_t^S\rangle$ and $P^S(x,t)$ are the state and the probability distribution for QWs with a SU(2) coin after t steps respectively. \square

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Corollary 2. *The average position is the same in quantum walks with a $U(2)$ coin and a $SU(2)$ coin if the two coins have the same parameters α , β and γ with the $U(2)$ coin.*

Proof. From Lemma 1, we can know that the average position for a $U(2)$ coin $\langle x \rangle = \sum_x x P(x, t) = \sum_x x P^S(x, t) \equiv \langle x \rangle^S$, where $\langle x \rangle^S$ is the average position for QWs with a $SU(2)$ coin. \square

With corollary 2, if we want to know the average position property in QWs with an arbitray unitary operator, we only need to study the quantum walk with a $SU(2)$ coin instead, for the rest part of this paper, we always use the $SU(2)$ coin as denoted in Eq. (6).

Following the analysis in Ref. [16], the state after t steps QWs with a $SU(2)$ coin

$$|\Psi_t\rangle = [S(I_P \otimes U_{\alpha, \beta, \gamma}^S)]^t |\Psi_0\rangle, \quad (8)$$

the spatial Fourier transformation for of the wave function $\Psi(x, t)$ over \mathcal{Z} is given by

$$\tilde{\Psi}(k, t) = \sum_{x=-\infty}^{\infty} \Psi(x, t) e^{ikx}, \quad (9)$$

where $k \in [-\pi, \pi]$. We can know

$$\tilde{\Psi}(k, t) = (M_k)^t \tilde{\Psi}(k, 0), \quad (10)$$

where

$$\begin{aligned} M_k &= e^{ik} M_+ + e^{-ik} M_- \\ &= \begin{pmatrix} e^{-i(k-\alpha)} \cos \beta, & -e^{-i(k+\gamma)} \sin \beta \\ e^{i(k+\gamma)} \sin \beta, & e^{i(k-\alpha)} \cos \beta \end{pmatrix}. \end{aligned} \quad (11)$$

The eigenvalues of M_k is

$$\begin{cases} \lambda_a = e^{-iw} \\ \lambda_b = e^{iw} \end{cases}, \quad (12)$$

where $\cos w = \cos(k - \alpha) \cos \beta$. And the eigenstates

$$\begin{cases} \tilde{\Psi}_k^a = \frac{1}{C_k^a} \begin{pmatrix} P_k \\ Q_k^a \end{pmatrix} \\ \tilde{\Psi}_k^b = \frac{1}{C_k^b} \begin{pmatrix} P_k \\ Q_k^b \end{pmatrix} \end{cases}, \quad (13)$$

where

$$\begin{cases} P_k = -e^{-i(k+\gamma)} \sin \beta \\ Q_k^a = -i \sin \omega_k + i \sin(k - \alpha) \cos \beta \\ Q_k^b = i \sin \omega_k + i \sin(k - \alpha) \cos \beta \end{cases}, \quad (14)$$

$$\begin{aligned} C_k^a &= \sqrt{P_k^* P_k + (Q_k^a)^* Q_k^a} \\ &= \sqrt{2(\sin^2 \omega_k - \cos \beta \sin(k - \alpha) \sin \omega_k)}, \end{aligned} \quad (15)$$

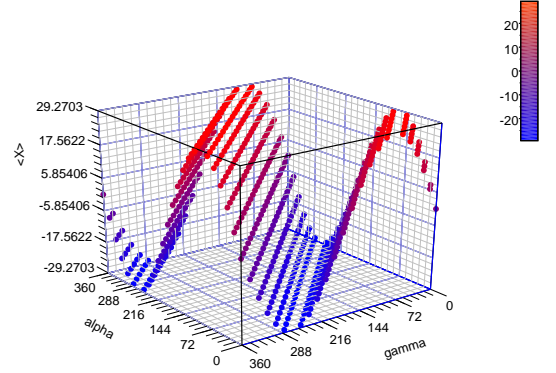


Figure 1. The average of position $\langle x \rangle$ for quantum walks after $t = 100$ steps with a $SU(2)$ coin, where $\beta = \pi/6$.

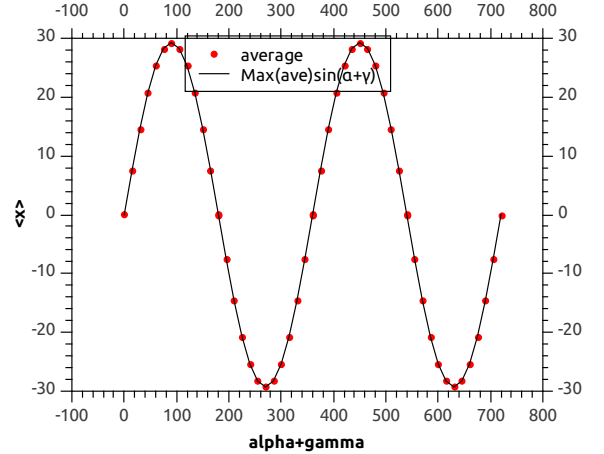


Figure 2. (red dot) The average of position with $\alpha + \gamma$ after 100 steps quantum walk when $\beta = \pi/6$, (line) $f(\phi) = \max(\langle x \rangle) \sin(\phi)$.

$$\begin{aligned} C_k^b &= \sqrt{P_k^* P_k + (Q_k^b)^* Q_k^b} \\ &= \sqrt{2(\sin^2 \omega_k + \cos \beta \sin(k - \alpha) \sin \omega_k)}. \end{aligned} \quad (16)$$

IV. THE AVERAGE POSITION IN QUANTUM WALKS

Fig. 1 and Fig. 2 show the average position after $t = 100$ steps QWs with a $SU(2)$ coin in the case of $\beta = \pi/6$, while the initial state is $1/\sqrt{2}(|0L\rangle + i|0R\rangle)$. From Fig. 1, we can know that $\langle x \rangle$ only depends on the sum of α and γ . Fig. 2 shows that the actual $\langle x \rangle$ exactly match the function of $f(\phi) = \max(\langle x \rangle) \sin(\phi)$, then we conject that $\langle x \rangle = G(\beta, t) \sin(\alpha + \gamma)$.

V. PROOF IN METHEMATICS

Theorem 3. *The probability distribution for quantum walks with a $SU(2)$ coin is independent on the parameter α and γ , when the initial state is $|0L\rangle$ or $|0R\rangle$. i.e.*

$$\begin{cases} P_{|0L\rangle}(\alpha, \beta, \gamma, x, t) = P_{|0L\rangle}(\beta, x, t) \\ P_{|0R\rangle}(\alpha, \beta, \gamma, x, t) = P_{|0R\rangle}(\beta, x, t) \end{cases}, \quad (17)$$

for any α and γ .

Proof. If the initial state $|\Psi_0\rangle = |0L\rangle$, then $\tilde{\Psi}(k, 0) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$. The probability of $|x\rangle$:

$$\begin{aligned} P_{|0L\rangle}(\alpha, \beta, \gamma, x, t) &= P_L(\alpha, \beta, \gamma, x, t) + P_R(\alpha, \beta, \gamma, x, t) \\ &= \frac{1}{4\pi^2} \int_{-\pi}^{\pi} \int_{-\pi}^{\pi} g_L(k_1, k_2, \alpha, \beta, \gamma, x, t) dk_1 dk_2 + \\ &\quad \frac{1}{4\pi^2} \int_{-\pi}^{\pi} \int_{-\pi}^{\pi} g_R(k_1, k_2, \alpha, \beta, \gamma, x, t) dk_1 dk_2, \end{aligned} \quad (18)$$

where $g_j(k_1, k_2, \alpha, \beta, \gamma, x, t) = \tilde{\Psi}_j^*(k_1, t) \tilde{\Psi}_j(k_2, t) e^{i(k_1 - k_2)x}$, $j \in \{L, R\}$. If we set $h = k - \alpha$, then we can know $g_j(k_1, k_2, \alpha, \beta, \gamma, x, t) = g_j(h_1, h_2, \beta, x, t)$, and $g_j(h_1, h_2, \beta, x, t) = g_j(h_1 \pm 2\pi, h_2, \beta, x, t) = g_j(h_1, h_2 \pm 2\pi, \beta, x, t)$, then

$$\begin{aligned} P_j(\alpha, \beta, \gamma, x, t) &= \frac{1}{4\pi^2} \int_{-\pi-\alpha}^{\pi-\alpha} \int_{-\pi-\alpha}^{\pi-\alpha} g_j(h_1, h_2, \beta, x, t) dh_1 dh_2 \\ &= \frac{1}{4\pi^2} \int_{-\pi}^{\pi} \int_{-\pi}^{\pi} g_j(h_1, h_2, \beta, x, t) dh_1 dh_2 \\ &= P_j(\beta, x, t). \end{aligned} \quad (19)$$

Further more we can know $P_{|0L\rangle}(\alpha, \beta, \gamma, x, t) = P_{|0L\rangle}(\beta, x, t)$. In the same way, we can also get $P_{|0R\rangle}(\alpha, \beta, \gamma, x, t) = P_{|0R\rangle}(\beta, x, t)$. \square

Theorem 4. *After t steps, we have*

$$\begin{cases} \Psi_{|0L\rangle}^R(x, t) + \Psi_{|0R\rangle}^L(-x, t) \in i\mathbb{R} \\ \Psi_{|0L\rangle}^R(x, t) - \Psi_{|0R\rangle}^L(-x, t) \in \mathbb{R} \end{cases}, \quad (20)$$

$$\begin{cases} \Psi_{|0L\rangle}^L(x, t) + \Psi_{|0R\rangle}^R(-x, t) \in \mathbb{R} \\ \Psi_{|0L\rangle}^L(x, t) - \Psi_{|0R\rangle}^R(-x, t) \in i\mathbb{R} \end{cases}, \quad (21)$$

where $\Psi_{|n\rangle}^m(x, t)$ denotes the coefficient of the $|x\rangle | m\rangle$ state after t steps quantum walk with the initial state $|n\rangle$.

Proof.

$$\begin{aligned} \Psi_{|0L\rangle}^R(x, t) \pm \Psi_{|0R\rangle}^L(-x, t) &= \frac{1}{2\pi} \int_{-\pi}^{\pi} \tilde{\Psi}_{|0L\rangle}^R(k, t) e^{-ikx} dk \pm \frac{1}{2\pi} \int_{-\pi}^{\pi} \tilde{\Psi}_{|0R\rangle}^L(k, t) e^{ikx} dk \\ &= \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{Q_k^a}{(C_k^a)^2} [P_k^* e^{-ikx} \mp P_k e^{ikx}] [e^{-i\omega_k t} - e^{i\omega_k t}] dk. \end{aligned} \quad (22)$$

As $Q_k^a \in i\mathbb{R}$, we can know $\Psi_{|0L\rangle}^R(x, t) + \Psi_{|0R\rangle}^L(-x, t) \in i\mathbb{R}$, and $\Psi_{|0L\rangle}^R(x, t) - \Psi_{|0R\rangle}^L(-x, t) \in \mathbb{R}$. Similarly, we can get Eq. (21). \square

Corollary 5. *The symmetry property of distribution between quantum walks with a $U(2)$ coin in initial state $|0L\rangle$ and $|0R\rangle$: For an arbitrary t , $P_{|0L\rangle}^R(\beta, x, t) = P_{|0R\rangle}^L(\beta, -x, t)$, $P_{|0L\rangle}^L(\beta, x, t) = P_{|0R\rangle}^R(\beta, -x, t)$.*

Proof. We set $\Psi_{|0L\rangle}^R(x) = C + Di$, where $C, D \in \mathbb{R}$. From Theorem 4, we can know $\Psi_{|0R\rangle}^L(-x) = -C + Di$, then we can know $P_{|0L\rangle}^R(\beta, x, t) = P_{|0R\rangle}^L(\beta, -x, t)$. Similarly, we can also get $P_{|0L\rangle}^L(\beta, x, t) = P_{|0R\rangle}^R(\beta, -x, t)$. \square

Theorem 6. *If the initial state $|\Psi_0\rangle = m|0L\rangle + n|0R\rangle$, where $|m|^2 + |n|^2 = 1$, the probability at state $|xL\rangle$ or $|xR\rangle$ after t steps quantum walk is*

$$\begin{cases} P^L(x) = |m|^2 P_{|0L\rangle}^L + |n|^2 P_{|0R\rangle}^L - (e^{-i(\alpha+\gamma)} m^* n + e^{i(\alpha+\gamma)} m n^*) G^L(\beta, x, t) \\ P^R(x) = |m|^2 P_{|0L\rangle}^R + |n|^2 P_{|0R\rangle}^R - (e^{-i(\alpha+\gamma)} m^* n + e^{i(\alpha+\gamma)} m n^*) G^R(\beta, x, t) \end{cases}, \quad (23)$$

where G^L and G^R are independent of α and γ .

Proof. The probability at state $|xL\rangle$ after t steps:

$$\begin{aligned} P_{m|0L\rangle+n|0R\rangle}^L(x) &= \frac{1}{4\pi^2} \int \int \tilde{\Psi}_L^*(k_1, t) \tilde{\Psi}_L(k_2, t) e^{i(k_1 - k_2)x} dk_1 dk_2 \\ &= |m|^2 P_{0L}^L(\beta, x, t) + |n|^2 P_{0R}^L(\beta, x, t) - \\ &\quad (e^{-i(\alpha+\gamma)} m^* n \sum_{i=1}^4 G_i + e^{i(\alpha+\gamma)} m n^* \sum_{i=1}^4 G_i^*), \end{aligned} \quad (24)$$

where

$$\begin{aligned} G_1(\beta, x, t) &= \frac{1}{4\pi^2} \int \int e^{i(\omega_{h_1} - \omega_{h_2})t} \frac{1}{(C_{h_1}^b C_{h_2}^b)^2} e^{i(h_1 - h_2)x} \sin^3 \beta e^{i(h_1 - h_2)x} e^{-ih_1} (Q_{h_2}^b)^* dh_1 dh_2 \\ &= \frac{1}{4\pi^2} \int \int \frac{(Q_{h_2}^b)^* \sin^3 \beta}{(C_{h_1}^b C_{h_2}^b)^2} i \sin[(\omega_{h_1} - \omega_{h_2}) + (h_1 - h_2) + (h_1 - h_2)x - h_1] dh_1 dh_2 \in R. \end{aligned} \quad (25)$$

As the same of $G_1(\beta, x, t)$, we can know $G_i(\beta, x, t) \in R$, where $i \in \{1, 2, 3, 4\}$. So Eq. (24) can be written as

$$P^L(x) = |m|^2 P_{|0L\rangle}^L + |n|^2 P_{|0R\rangle}^L - (e^{-i(\alpha+\gamma)} m^* n + e^{i(\alpha+\gamma)} mn^*) G^L(\beta, x, t), \quad (26)$$

where $G^L(\beta, x, t) = \sum_{i=1}^4 G_i$. In the same way as $P^L(x)$, we can get $P^R(x)$ in Eq. (23). \square

Theorem 7. *If the initial state $|\Psi_0\rangle = 1/\sqrt{2}(|0L\rangle + i|0R\rangle)$, The average position after t steps quantum walk: $\langle x \rangle = G(\beta, t) \sin(\alpha + \gamma)$, where $G(\beta, t)$ only depends on β and t .*

Proof. From Corollary 5, we can know

$$\begin{cases} \sum_{x=-t}^t x [P_{|0L\rangle}^R(\beta, x, t) + P_{|0R\rangle}^L(\beta, x, t)] = 0 \\ \sum_{x=-t}^t x [P_{|0L\rangle}^L(\beta, x, t) + P_{|0R\rangle}^R(\beta, x, t)] = 0 \end{cases} \quad (27)$$

Using Eq. (27) and Theorem 6 we can know

$$\begin{aligned} \langle x \rangle &= \sum_{x=-t}^t x (P^L(\alpha, \beta, \gamma, x, t) + P^R(\alpha, \beta, \gamma, x, t)) \\ &= G(\beta, t) \sin(\alpha + \gamma), \end{aligned} \quad (28)$$

where $G(\beta, t) = -\sum_{x=-t}^t x [G^L(\beta, x, t) + G^R(\beta, x, t)]$ only depends on β and t , regardless of α or γ . \square

VI. CONCLUSIONS

In this paper, we discussed the properties of the average position in QWs with an arbitrary unitary coin. With a SU(2) coin, if the initial state is $|0L\rangle$ or $|0R\rangle$, the probability distribution is independent on α and γ . Some symmetry properties between different initial states $|0L\rangle$ and $|0R\rangle$ was proved, we get that $P_{|0L\rangle}^R(\beta, x, t) = P_{|0R\rangle}^L(\beta, -x, t)$ and $P_{|0L\rangle}^L(\beta, x, t) = P_{|0R\rangle}^R(\beta, -x, t)$. If the initial state $|\Psi_0\rangle = 1/\sqrt{2}(|0L\rangle + i|0R\rangle)$, we can know the average $\langle x \rangle = G(\beta, t) \sin(\alpha + \gamma)$, so if we replace the Hadamard operator with an arbitrary unitary operator, the average position is always not equal to 0, unless $\alpha + \beta = n\pi, n \in \mathbb{Z}$.

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- [1] Y. Aharonov, L. Davidovich, and N. Zagury, Phys. Rev. A **48**, 1687 (1993).
 - [2] E. Farhi, and S. Gutmann, Phys. Rev. A **58**, 915 (1998)
 - [3] A. M. Childs, Phys. Rev. Lett. **102**, 180501 (2009).
 - [4] N. B. Lovett, S. Cooper, M. Everitt, M. Trevers, and V. Kendon, Phys. Rev. A **81**, 042330 (2010)
 - [5] A. M. Childs, R. Cleve, E. Deotto, E. Farhi, S. Gutmann, and D. A. Spielman, Proceedings of the 35th ACM symposium on Theory of computing (ACM Press, New York), (2003), pp. 59-68.
 - [6] N. Shenvi, J. Kempe, and K. Birgitta Whaley, Phys. Rev. A **67**, 052307 (2003).
 - [7] A. M. Childs, E. Farhi, and S. Gutmann, Quantum Information Processing, Vol. 1, pp. 35-43 (2002).
 - [8] A. M. Childs and J. Goldstone, Phys. Rev. A **70**, 022314 (2004).
 - [9] A. Ambainis, arXiv:quant-ph/0403120.
 - [10] A. Ambainis and J. Kempe, In Proceedings of the 16th ACM-SIAM symposium on Discrete algorithms, pp. 1099-1108, (2005).
 - [11] D. Aharonov, A. Ambainis, J. Kempe, U. Vazirani, Proceeding STOC '01 Proceedings of the thirty-third annual ACM symposium on Theory of computing, pp. 50 - 59 (2001).
 - [12] L. C. Kwek and Setiawan, Phys. Rev. A **84**, 032319 (2011).
 - [13] T. A. Brun, H. A. Carteret, and A. Ambainis, Phys. Rev. A **67**, 052317 (2003).
 - [14] T. A. Brun, H. A. Carteret, and A. Ambainis, Phys. Rev. A **67**, 032304 (2003).
 - [15] C. M. Chandrashekar, R. Srikanth, and R. Laflamme, Phys. Rev. A **77**, 032306 (2008).
 - [16] A. Nayak and A. Vishwanath, Technical Report, Center for Discrete Mathematics & Theoretical Computer Science (2000).